# RA 20

## 10.9 Counting Multisets

- A multiset is a collection that can have multiple instances of the same item.

  - Rules for encoding a selection of n objects from m variables:

| Selection | Code Words |
|---|---|
| $n =$ # of items to select | $n =$ number of 0s in code word |
| $m =$ # of varieties | $m-1 =$ # of 1's in code word. |
| # selected from the $1^{st}$ variety | # of zeros before the first 1 |
| # selected from $i^{th}$ variety, for $1 < i < m$ | # of zeros between the $i-1^{th}$ and $i^{th}$ 1 |
| # selected from the last variety | # of 0s after last 1 |

- The # of ways to select n objects from m varieties is
  ↳ $\binom{n+m-1}{m-1}$

if there is no limitation on the # of each variety available and objects of the same variety are indistinguishable

- Example : $n=12$ cookies $m=4$ flavors $\rightarrow \binom{15}{3}$

- The # of ways to place n distinguishable balls into m distinguishable bins is:
  ↳ $\binom{n+m-1}{m-1}$

## Examples

- Bakery sells 7 varieties, How many ways to select 12 donuts?

  $\binom{12+7-1}{7-1} = \binom{18}{6}$

- Bakery sells 7 varieties, How many ways to select 12 donuts if the selection must have at least one variety?

  $\hookrightarrow \binom{5+7-1}{7-1} = \binom{11}{6}$

## 10.10 Assignment problems: Balls to bins

| | No Restrictions | At most one ball per bin | Same # of balls in each bin |
|---|---|---|---|
| | (Any positive m and n) | (m must be at least n) | (m must evenly divide n) |
| Indistinguishable Balls | $\binom{n+m-1}{m-1}$ | $\binom{m}{n}$ | 1 |
| Distinguishable Balls | $m^n$ | $P(m,n)$ | $\frac{n!}{\left(\left(\frac{n}{m}\right)!\right)^m}$ |

## 10.11 Inclusion - Exclusion Principle

- The <u>principle of inclusion-exclusion</u> is a technique for determining the cardinality of the union of sets that uses the cardinality of each individual set and the cardinality of the intersections of two sets.

- The inclusion - exclusion principle with two sets
  - Let A and B be 2 finite sets, then $|A \cup B| = |A| + |B| - |A \cap B|$

- The inclusion - exclusion principle with 3 sets
  - Let A, B, C, be 3 finite sets then:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$

- The general inclusion - exclusion principle applied to 4 sets

$|A \cup B \cup C \cup D|$

$= |A| + |B| + |C| + |D|$ (add the sizes of the sets)

$- |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D|$ (minus pairwise intersection)

$+ |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D|$ (plus 3way intersections)

$- |A \cap B \cap C \cap D|$ (minus 4-way intersections)

- The inclusion - exclusion principle and the sum rule
  - A collection of sets is <u>mutually disjoint</u> if the intersection of every pair of sets in the collection is empty
    - in this case $|A_1 \cup A_2 \cup .. \cup A_n| = |A_1| + |A_2| + ... + |A_n|$

- You can also determine the cardinality of a union by complement:

$$|U| - |\overline{P_1 \cup P_2 \cup P_3 \cup ... \cup P_n}| = |P_1 \cup P_2 \cup ... \cup P_n|$$